The good the bad and the ugly in differential calculus.

The following are a group of Questions most of them Question 6c or Question 7c on paper 1 of the higher Leaving Cert maths which most students found very difficult, the purpose of these questions was to filter out the A1, and A2 students from the rest.

Example 1

Question 7c leaving Cert Higher Maths 2000.

If $f(x)=\frac{\ln x}{x}$ show the max value of f(x) occurs at (e,1/e).

This is a straightforward max/min problem, we know the max or min occurs when dy/dx = 0, so find dy/dx set it equal to 0 and solve for x.

$$f(x)=\frac{\ln x}{x}\Rightarrow f'(x)=\frac{x.\frac{1}{x}-\ln x(1)}{x^2}=\frac{1-\ln x}{x^2}=0\Rightarrow$$
$$1-\ln x=0\Rightarrow \ln x=1\Rightarrow x=e, y=\frac{\ln e}{e}=1/e$$

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$$f''(x)=\frac{x^2(-1/x)-(1-\ln x)2x}{x^4} @ x=$$
$$e, f''(x)=\frac{e^2(-1/e)-0}{e^4}=-1/e^3<0\Rightarrow$$
. max at (e,1/e). This was worth 10 marks

It was the second part of this question, which for most students was a total write off.

It said ***hence*** show $x^e \le e^x$. Most students did not have a clue!

Solution:

We know from above that the maximum value of

$$\frac{\ln x}{x} = 1/e \Rightarrow \frac{\ln x}{x} \le 1/e \Rightarrow e\ln x \le x \Rightarrow$$

$$**\ln x^e \le x \Rightarrow ***x^e \le e^x$$

Here we use the rule $x\ln a = \ln a^x$, * we use the rule $\log_b a = x \Rightarrow a = b^x$.

This next question we will look at was asked in 1997 in fact it is one of two questions, which were really over the top on the 1997 paper.

Question 6c 1997 paper 1.

If $Siny = \frac{1}{2}(1-x^2)$ find the value of a and the value of b if $(\frac{dy}{dx})^2 = \frac{a}{3-x^2} - \frac{b}{1+x^2}$. There are many ways to do this, this is one of the better ways, treat Siny as an implicit function

$$Cosy\frac{dy}{dx} = -x \Rightarrow \frac{dy}{dx} = \frac{-x}{Cosy} \Rightarrow (\frac{dy}{dx})^2 = \frac{x^2}{(Cosy)^2}$$

$$= \frac{x^2}{1-Sin^2x} = \frac{x^2}{(1-Siny)(1+Siny)}$$

$$1+Siny = 1+\frac{1}{2}(1-x^2) = \frac{3}{2}-\frac{x^2}{2} = \frac{3-x^2}{2} \quad 1-Siny =$$

$$1-\frac{1}{2}(1-x^2) = \frac{1}{2}+\frac{x^2}{2} = \frac{1+x^2}{2} \Rightarrow$$

$$(\frac{dy}{dx})^2 = \frac{x^2}{(\frac{3-x^2}{2})(\frac{1+x^2}{2})} = \frac{4x^2}{(3-x^2)(1+x^2)} =$$

$$\frac{a(1+x^2)-b(3-x^2)}{(3-x^2)(1+x^2)} \Rightarrow *a-3b = 0, a+b = 4$$

*Set the top lines equal to each other this gives a = 3, b = 1.

You can see from the above that this was a lot of hardship for 10 marks .

The second question on the 1997 was as follows :

Question 7 ©1997 Paper 1.

Let $y = x - 1 + \frac{1}{x-1}$ (i) find the values of x for which dy/dx = 0.

(ii)for x real show that y cannot have a real value between -2 and 2.

Solution:

$$\frac{dy}{dx} = 1 - \frac{1}{(x-1)^2} = 0 \Rightarrow (x-1)^2 = 1 \Rightarrow x^2 - 2x + 1 = 1$$
$$\Rightarrow x^2 - 2x \Rightarrow x(x-2) = 0 \Rightarrow x = 0, x = 2$$

(ii) this part again proved to be very difficult, the key is the words "for x real"

So turn the equation from part (i) into a quadratic by multipling everything by (x-1).

$$y(x-1) = (x-1)^2 + 1 \Rightarrow xy - y = x^2 - 2x + 1 + 1 \Rightarrow$$
$$x^2 + x(-2-y) + 2 + y = 0,$$
$$Roots..are..real..when..b^2 - 4ac \geq 0 \Rightarrow$$
$$(-2-y)^2 - 4(2+y) \geq 0 \Rightarrow y^2 - 4 \geq 0 \Rightarrow -2 \geq y \geq 2$$

that is y cannot lie between -2 and 2.

In 1996 Question 6c (i) they asked this little gem !

If $x = a(\theta + Sin\,\theta) : y = a(1 - Cos\,\theta)$ if a is a constant show that $1 + (\frac{dy}{dx})^2 = Sec^2\frac{\theta}{2}$.

This was a question based on parametric differentiation that is $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ do this first

$$\frac{dy}{d\theta} = a + aCos\,\theta .. \frac{dx}{d\theta} = aSin\,\theta \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a + aCos\,\theta}{aSin\,\theta} = \frac{1 + Cos\,\theta}{Sin\,\theta}.$$

Many students got this far but getting from here to the end proved very difficult and frankly was not worth the bother since it was worth at most 5 marks!

The trick was to realise that you needed $\frac{\theta}{2}$ .To change from $\theta..to..\frac{\theta}{2}$ we use Page 9 of the tables . $Sin2A = \frac{2\tan A}{1+\tan^2 A}$. $Cos2A = \frac{1-\tan^2 A}{1+\tan^2 A}$ use the corresponding connection between A and A/2,

if $Tan\frac{\theta}{2} = t \Rightarrow \frac{dy}{dx} = \frac{1+\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = t \Rightarrow 1+(\frac{dy}{dx})^2 = 1+t^2 = 1+\tan^2\frac{\theta}{2} = \sec^2\frac{\theta}{2}$.

The problem here was that most students subbed in $\frac{1+\cos\theta}{\sin\theta}$ into $1+(\frac{dy}{dx})^2$ and could not simplify it down !

Question 7c on the 1995 paper 1 was another question sent to try us.

It says let $x = \frac{1}{2}(e^y - e^{-y})$ show $y = \ln(x+\sqrt{x^2+1})$. This looked worse than it actually was it really was just an index equation .

$$x = \frac{1}{2}(e^y - e^{-y}) \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y =$$

$$(e^y)^2 - 1,.let..e^y =$$

$$a \Rightarrow 2xa = a^2 - 1 \Rightarrow a^2 - 2ax - 1 = 0$$

$$\Rightarrow a = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow a = x \pm \sqrt{x^2+1} \Rightarrow a =$$

$$x + \sqrt{x^2+1} ** \Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow y = \ln(x + \sqrt{x^2+1})$$

**We use the “+” form as We can only find ln of a positive number .

Since $x - \sqrt{x^2+1} < 0$.

The Second part asked us to find dy/dx in the form $\frac{p}{(1+x^2)^q}$

$$y = \ln(x + \sqrt{x^2+1}) \Rightarrow \frac{dy}{dx} =$$

$$\frac{1}{x+\sqrt{x^2+1}}(1+\tfrac{1}{2}(x^2+1)^{\frac{-1}{2}}2x) =$$

$$\frac{1}{x+\sqrt{x^2+1}}(1+\frac{x}{\sqrt{x^2+1}}) =$$

$$\frac{1}{x+\sqrt{x^2+1}}(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}) = \frac{1}{\sqrt{x^2+1}} = \frac{1}{(x^2+1)^{\frac{1}{2}}}$$

The algebra at the end of this was nice .

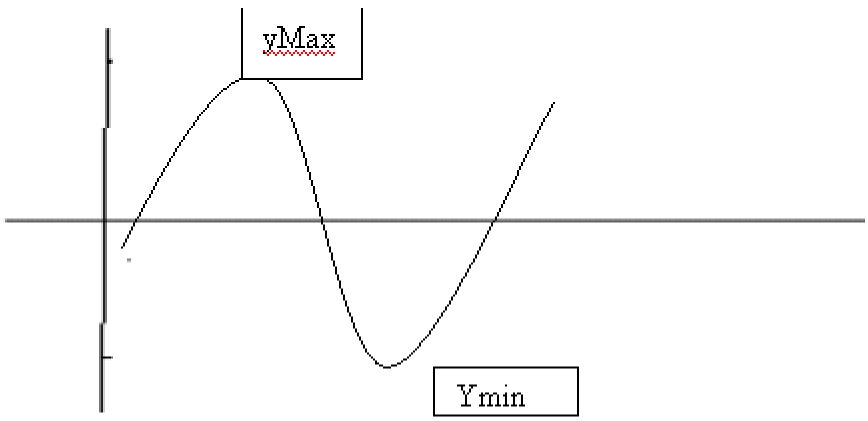
[Question 7c 1999 paper1]:

$f(x) = x^3 + kx^2 - 4$ show f(x) has a local minimum at(0,-4) and a local maximum at

$\{ \frac{-2k}{3}, \frac{4k^3-108}{27})$.

To find the Max and Min, just find dy/dx, set dy/dx = 0

$$f'(x) = 3x^2 + 2kx = 0 \Rightarrow x(3x+2k) = 0 \Rightarrow x = 0, x = -2k/3$$
$$f''(x) = 6x + 2k, @\, x = 0. f''(0) = 2k > 0 \Rightarrow Min\, @\, x = 0.$$
$$f''(-2k/3) = -2k < 0 \Rightarrow Max\, @\, x = -2k/3.$$
$$x = 0, f(0) = -4$$
$$x = (-2k/3) \Rightarrow f(-2k/3) = (-2k/3)^3 + 2k(-2k/3)^2 - 4 = \frac{4k^3 - 108}{27}$$



The next part ask to find the range of values of k for which f(x)=0 has 3 real roots.

If f(x) has 3 real roots (Ymax)(Ymin)<0

$$-4\left(\frac{4k^3 - 108}{27}\right) < 0 \Rightarrow k > 3$$

If f(x) has two equal roots then Ymax.Ymin = 0 k= 3.